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Selection Rules of Energy-Level Transition for the Capacitance Coupling LC Mesoscopic Circuit by Using Invariant Eigen-Operator Method

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Abstract This paper examines the quantization for the capacitance coupling inductance-capacitance (LC) mesoscopic circuit. By virtue of the "invariant eigen-operator" (IEO) method, it deduces the selection rules of energy-level transition when the system is disturbed by the external electromagnetic field.

Keywords Mesoscopic · Invariant eigen-operator · Selection rules · Energy-level transition

1 Introduction

Due to the rapid progress in quantum computation and quantum information, a substantial interest has revived the physical research on the quantum systems [1–9]. Among all kinds of possible scheme for quantum computation realization, because of easy integration, the small solid-state devices are promising for quantum circuit integration for quantum computation. So it is of importance to investigate the mesoscopic circuit including the elementary components such as capacitance, inductance and electric resistance etc. A single inductance-capacitance (LC) nondissipative mesoscopic circuit is one fundamental cell of mesoscopic electric circuits. Its quantization and quantum effects were first discussed by Louisell [10] and some more progress has been made in this field in recent years [11–14]. But most of work paid more attention to the quantization and the quantum effects. In fact, in order to study the system fully, it is also of significance to obtain the energy-level information. In this paper, we shall investigate, if a capacitance coupling inductance-capacitance (LC) mesoscopic circuit is disturbed, e.g., radiated by a external electromagnetic field, what selection rules are followed when the system transits from a energy-level to another.

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Our procedures are as follows: firstly, for convenience, we introduce the "invariant eigen-operator" (IEO) method. Secondly, we give the quantum-mechanical Hamiltonian of this system. Finally, by virtue of IEO method, we obtain the selection rules of energy-level transition when the system is disturbed.

2 The "Invariant Eigen-Operator" Method

Several years ago, a new method named "invariant eigen-operator" (IEO) method was suggested [15–17], steaming from the Heisenberg equation of motion. The Heisenberg equation of motion is

$$\frac{d}{dt}\hat{O} = \frac{1}{i\hbar}[\hat{O}, \hat{\mathcal{H}}], \tag{1}$$

which is of the same importance as the Schrödinger equation. However, since (1) does not involve wavefunctions or eigenvectors, it can hardly be directly employed to derive energy-level formulas. Provided that we can find such a operator which satisfies the following eigenvector-like equation

$$\left(i\hbar\frac{d}{dt}\right)^2 \hat{O}_e = \lambda \hat{O}_e, \quad \lambda > 0, \tag{2}$$

the Heisenberg equation can also be used to deduce energy levels of certain systems in a direct way. Here, it mean the operator $\hat{O}_e$ is "invariant" under action of $(i\hbar\frac{d}{dt})^2$. By virtue of (1), we see

$$\left(i\hbar\frac{d}{dt}\right)^2 \hat{O}_e = [[\hat{O}_e, \hat{\mathcal{H}}], \hat{\mathcal{H}}] = \lambda \hat{O}_e. \tag{3}$$

From this equation we can judge that $\sqrt{\lambda}$ is related to energy eigenvalues of $\hat{\mathcal{H}}$. This is because, as Schrödinger initiated, $i\hbar\frac{d}{dt} \leftrightarrow \hat{\mathcal{H}}$, $(i\hbar\frac{d}{dt})^2 \leftrightarrow \hat{\mathcal{H}}^2$, (3) can be looked on as a parallel equation with the energy eigen-vector equation $\hat{\mathcal{H}}^2\Psi = E^2\Psi$. Thus, once the invariant eigen-operator $\hat{O}_e$ is found, some information about energy eigenvalues can be obtained.

3 Hamiltonian for the Capacitance Coupling LC Mesoscopic Circuit

The capacitance coupling LC mesoscopic circuits are drawn in Fig. 1, where L_j $(j = 1, 2)$ is the self-inductance coefficient of the jth inductance, C_c is the coupling capacitance and

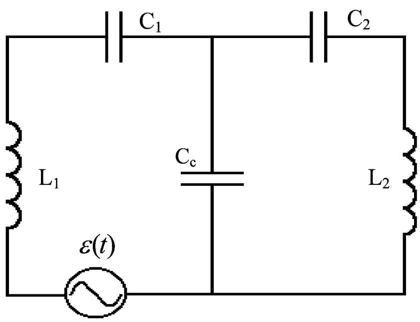

Fig. 1 The capacitance coupling LC mesoscopic circuit

C_j is the capacitance of the jth capacitor. Here, supposed that the circuit is excited by an instant impulse source. If we consider q_j, which represents charge for the jth $(j = 1, 2)$ branch circuit, as the generalized coordinate, the classical Lagrangian of the system is

$$\mathcal{L} = \frac{1}{2}L_1\dot{q}_1^2 + \frac{1}{2}L_2\dot{q}_2^2 - \frac{(C_c + C_1)q_1^2}{2C_cC_1} - \frac{(C_c + C_2)q_2^2}{2C_cC_2} + \frac{q_1q_2}{C_c}, \tag{4}$$

from which the conjugate momenta p_1 and p_2 are given by

$$p_1 \equiv \frac{\partial\mathcal{L}}{\partial\dot{q}_1} = L_1\dot{q}_1, \qquad p_2 \equiv \frac{\partial\mathcal{L}}{\partial\dot{q}_2} = L_2\dot{q}_2. \tag{5}$$

Note that p_j $(j = 1, 2)$ is the self-inductance magnetic flux through the jth inductance.

Thus, we can deduce the classical Hamiltonian of the system

$$\begin{aligned}\mathcal{H} &= p_1\dot{q}_1 + p_2\dot{q}_2 - \mathcal{L} \\ &= \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} + \frac{1}{2}m_1\omega_1^2q_1^2 + \frac{1}{2}m_2\omega_2^2q_2^2 - \frac{q_1q_2}{C_c},\end{aligned} \tag{6}$$

where $m_j = L_j$ and $\omega_j \equiv \sqrt{\frac{C_j+C_c}{L_jC_jC_c}}$. According to the standard canonical quantization principle, a pair of canonical conjugate quantities p_j and q_j is associated with a pair of Hermitian operators $\hat{p}_j$ and $\hat{q}_j$. They satisfy the commutation relation $[\hat{q}_j, \hat{p}_j] = i\hbar$. So the quantized Hamiltonian operator of the system can be written as

$$\hat{\mathcal{H}} = \frac{\hat{p}_1^2}{2m_1} + \frac{\hat{p}_2^2}{2m_2} + \frac{1}{2}m_1\omega_1^2\hat{q}_1^2 + \frac{1}{2}m_2\omega_2^2\hat{q}_2^2 - \frac{\hat{q}_1\hat{q}_2}{C_c}. \tag{7}$$

In order to diagonalize the operator $\hat{\mathcal{H}}$, introduce the following unitary operator [18]

$$\hat{U} \equiv \int_{-\infty}^{+\infty}\int dq_1dq_2 \left|\begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}\right\rangle\left\langle\begin{pmatrix} q_1 \\ q_2 \end{pmatrix}\right|, \tag{8}$$

where A, B, C and D are the real number to be determined, which satisfy the relation $AD - BC = 1$, $|\binom{q_1}{q_2}\rangle = |q_1\rangle|q_2\rangle$ is two-mode coordinate eigenstate, and

$$|q_j\rangle = \left(\frac{m_j\omega_j}{\pi\hbar}\right)^{1/4}\exp\left(-\frac{m_j\omega_j}{2\hbar}q_j^2 + \sqrt{\frac{2m_j\omega_j}{\hbar}}q_j\hat{a}_j^\dagger - \frac{a_j^{\dagger 2}}{2}\right)|0\rangle. \tag{9}$$

Here, we assign

$$A = (L_2/L_1)^{1/4}\cos\frac{\varphi}{2}, \qquad B = -(L_2/L_1)^{1/4}\sin\frac{\varphi}{2}, \tag{10}$$

$$C = (L_1/L_2)^{1/4}\sin\frac{\varphi}{2}, \qquad D = (L_1/L_2)^{1/4}\cos\frac{\varphi}{2}, \tag{11}$$

$$\tan\varphi = 2\sqrt{L_1L_2}\left[L_1\left(1 + \frac{C_c}{C_2}\right) - L_2\left(1 + \frac{C_c}{C_1}\right)\right]^{-1}. \tag{12}$$

From (8) the following relations can be deduced

$$\hat{U}^{-1}\hat{q}_1\hat{U} = A\hat{q}_1 + B\hat{q}_2, \qquad \hat{U}^{-1}\hat{q}_2\hat{U} = C\hat{q}_1 + D\hat{q}_2, \tag{13}$$

$$\hat{U}^{-1}\hat{p}_1\hat{U} = D\hat{p}_1 - C\hat{p}_2, \qquad \hat{U}^{-1}\hat{p}_2\hat{U} = -B\hat{p}_1 + A\hat{p}_2. \tag{14}$$

So,

$$\begin{aligned}\hat{\mathcal{H}}' &= \hat{U}^{-1}\hat{\mathcal{H}}\hat{U} \\ &= \frac{1}{2m_1'}\hat{p}_1^2 + \frac{1}{2m_2'}\hat{p}_2^2 + \frac{1}{2}m_1'\omega_1'^2\hat{q}_1^2 + \frac{1}{2}m_2'\omega_2'^2\hat{q}_2^2,\end{aligned} \tag{15}$$

where

$$m_1' = m_2' = \sqrt{L_1L_2}, \tag{16}$$

$$\omega_1'^2 = \frac{1}{\sqrt{L_1L_2}}\left[m_1\omega_1^2A^2 + m_2\omega_2^2C^2 - \frac{AC}{C_c}\right], \tag{17}$$

$$\omega_2'^2 = \frac{1}{\sqrt{L_1L_2}}\left[m_1\omega_1^2B^2 + m_2\omega_2^2D^2 - \frac{BD}{C_c}\right]. \tag{18}$$

From the commutation relation $[\hat{q}_j, \hat{p}_j] = i\hbar$ we can construct the following Bosonic operators

$$\hat{a}_j = \sqrt{\frac{m_j'\omega_j'}{2\hbar}}\left[\hat{q}_j + \frac{i}{m_j'\omega_j'}\hat{p}_j\right], \tag{19}$$

$$\hat{a}_j^\dagger = \sqrt{\frac{m_j'\omega_j'}{2\hbar}}\left[\hat{q}_j - \frac{i}{m_j'\omega_j'}\hat{p}_j\right], \tag{20}$$

which satisfy the following relations

$$\left[\hat{a}_j, \hat{a}_l^\dagger\right] = \delta_{jl}, \qquad \left[\hat{a}_j, \hat{a}_l\right] = \left[\hat{a}_j^\dagger, \hat{a}_l^\dagger\right] = 0. \tag{21}$$

Substituting (19) and (20) into (15) leads to

$$\hat{\mathcal{H}}' = \hbar\omega_1'\left(\hat{a}_1^\dagger\hat{a}_1 + \frac{1}{2}\right) + \hbar\omega_2'\left(\hat{a}_2^\dagger\hat{a}_2 + \frac{1}{2}\right) \equiv \hat{\mathcal{H}}_1' + \hat{\mathcal{H}}_2'. \tag{22}$$

Equation (22) means that the quantized capacitance coupling two LC mesoscopic circuit, after a unitary transformation, can be considered as two independent quantum mechanical harmonic oscillators. Thus, the system is equivalent with a non-coupling two-mode radiation field. Generally, a system will stay at a certain stationary state if it is not affected by an external disturbance (say, a external electromagnetic field). But a practical system always interacts with external field, which leads to the system's transition from a stationary state to another. Dirac [19] pointed out that if, in the Heisenberg representation including the two stationary states serving as its two base vectors, the matrix element, which is related to the two states in the expression of the system's total electric displacement, is equal to zero, it is impossible to emerge the transition between the two states when disturbed by the electromagnetic radiation. In the next section we shall investigate the selection rules of energy-level transition for the capacitance coupling LC mesoscopic circuit by virtue of IEO method.

4 Deriving the IEO and the Selection Rules of Energy-Level Transition for the System

From the following commutation relations

$$[\hat{a}_1, \hat{\mathcal{H}}'] = [\hat{a}_1, \hat{\mathcal{H}}_1'] = \hbar\omega_1' \hat{a}_1, \qquad [\hat{a}_1^\dagger, \hat{\mathcal{H}}'] = [\hat{a}_1^\dagger, \hat{\mathcal{H}}_1'] = -\hbar\omega_1' \hat{a}_1^\dagger, \tag{23}$$

$$[\hat{a}_2, \hat{\mathcal{H}}'] = [\hat{a}_2, \hat{\mathcal{H}}_2'] = \hbar\omega_2' \hat{a}_2, \qquad [\hat{a}_2^\dagger, \hat{\mathcal{H}}'] = [\hat{a}_2^\dagger, \hat{\mathcal{H}}_2'] = -\hbar\omega_2' \hat{a}_2^\dagger, \tag{24}$$

we see that the invariant eigen-operators of the system are

$$\hat{O}_{e1} = \hat{a}_1, \qquad \hat{O}_{e1}' = \hat{a}_1^\dagger, \qquad \hat{O}_{e2} = \hat{a}_2, \qquad \hat{O}_{e2}' = \hat{a}_2^\dagger, \tag{25}$$

which satisfy the relations

$$\left(i\hbar\frac{d}{dt}\right)^2 \hat{a}_1 = [[\hat{a}_1, \hat{\mathcal{H}}'], \hat{\mathcal{H}}'] = \hbar^2\omega_1'^2 \hat{a}_1, \tag{26}$$

$$\left(i\hbar\frac{d}{dt}\right)^2 \hat{a}_1^\dagger = [[\hat{a}_1^\dagger, \hat{\mathcal{H}}'], \hat{\mathcal{H}}'] = \hbar^2\omega_1'^2 \hat{a}_1^\dagger, \tag{27}$$

$$\left(i\hbar\frac{d}{dt}\right)^2 \hat{a}_2 = [[\hat{a}_2, \hat{\mathcal{H}}'], \hat{\mathcal{H}}'] = \hbar^2\omega_2'^2 \hat{a}_2, \tag{28}$$

$$\left(i\hbar\frac{d}{dt}\right)^2 \hat{a}_2^\dagger = [[\hat{a}_2^\dagger, \hat{\mathcal{H}}'], \hat{\mathcal{H}}'] = \hbar^2\omega_2'^2 \hat{a}_2^\dagger. \tag{29}$$

Supposing $|c\rangle$ and $|b\rangle$ are two arbitrary eigenstates of the Hamiltonian $\hat{\mathcal{H}}'$, respectively, with eigenvalues E_c and E_b, from (26)–(29), we obtain

$$\left[(E_b - E_c)^2 - \hbar^2\omega_1'^2\right] \langle c| \hat{a}_1 |b\rangle = 0, \tag{30}$$

$$\left[(E_b - E_c)^2 - \hbar^2\omega_1'^2\right] \langle c| \hat{a}_1^\dagger |b\rangle = 0, \tag{31}$$

$$\left[(E_b - E_c)^2 - \hbar^2\omega_2'^2\right] \langle c| \hat{a}_2 |b\rangle = 0, \tag{32}$$

$$\left[(E_b - E_c)^2 - \hbar^2\omega_2'^2\right] \langle c| \hat{a}_2^\dagger |b\rangle = 0. \tag{33}$$

Using (30) and (31) we deduce

$$\left[(E_b - E_c)^2 - \hbar^2\omega_1'^2\right] \langle c| \overrightarrow{D}_1 |b\rangle = 0, \tag{34}$$

where

$$\begin{aligned} \overrightarrow{D}_1 &= \varepsilon_1 \overrightarrow{E}_1 \\ &= i\varepsilon_1 \left(\frac{\hbar\omega_1'}{2\varepsilon_1}\right)^{1/2} \left[\hat{a}_1 \vec{u}_1(\vec{r}_1)\, e^{-i\omega_1' t} - \hat{a}_1^\dagger \vec{u}_1^*(\vec{r}_1)\, e^{i\omega_1' t}\right] \end{aligned} \tag{35}$$

is the electric displacement vector operator [20] of the field represented by the Hamiltonian operator $\hat{\mathcal{H}}_1'$, $\vec{u}_1$ is the vector mode function which corresponds to the frequency ω_1', and ε_1, the dielectric constant. Similarly, we obtain

$$\left[(E_b - E_c)^2 - \hbar^2\omega_2'^2\right] \langle c| \overrightarrow{D}_2 |b\rangle = 0, \tag{36}$$

where

$$\begin{aligned}\vec{D}_2 &= \varepsilon_2 \vec{E}_2 \\ &= i\varepsilon_2 \left(\frac{\hbar\omega_2'}{2\varepsilon_2}\right)^{1/2} \left[\hat{a}_2 \vec{u}_2(\vec{r}_2) e^{-i\omega_2' t} - \hat{a}_2^\dagger \vec{u}_2^*(\vec{r}_2) e^{i\omega_2' t}\right]\end{aligned} \tag{37}$$

is the electric displacement vector operator of the field represented by the Hamiltonian operator $\hat{\mathcal{H}}_2'$. In the Heisenberg representation, if the matrix element of the operator $\vec{D}$, is equal to zero, it is impossible to emerge the transition between the two states related to the matrix element, as is mentioned above. Thus, the possibility of energy-level transition is restricted closely. As far as (34) and (36) is concerned, $\langle c|\vec{D}_j|b\rangle = 0$ $(j = 1, 2)$ except that $(E_b - E_c)^2 - \hbar^2\omega_j'^2 = 0$, i.e., the energy-level transition only possibly exists in between the two states whose energy-level difference satisfies the following relations

$$E_b - E_c = \pm\hbar\omega_j', \quad (j = 1, 2), \tag{38}$$

which gives the selection rules of energy-level transition when the system is disturbed. Because a unitary transformation can't alter the eigenvalues of the system, (38) also gives the selection rules of energy-level transition between $|\psi_1\rangle = \hat{U}|b\rangle$ and $|\psi_2\rangle = \hat{U}|c\rangle$ for the capacitance coupling LC mesoscopic circuit. When the transition occurs, the system will absorb or emit a phonon with the frequency $\Omega_1 = \omega_1'$ or $\Omega_2 = \omega_2'$.

In summary, by using the "invariant eigen-operator" method we conveniently obtain the selection rules of energy-level transition for the capacitance coupling two LC mesoscopic circuit, which enriches the theoretical research on mesoscopic circuit from a fresh point of view.

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